EQUIVALENCE RELATIONS ON SEPARATED NETS ARISING FROM LINEAR TORAL FLOWS

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Abstract. In 1998, Burago-Kleiner and McMullen independently proved the existence of separated nets in $\mathbb{R}^d$ which are not bi-Lipschitz equivalent (BL) to a lattice. A finer equivalence relation than BL is bounded displacement after dilation (BDD). Separated nets arise naturally as return times to a section for minimal $\mathbb{R}^d$-actions. We analyze the separated nets which arise via these constructions, focusing particularly on nets arising from linear $\mathbb{R}^d$-actions on tori. We show that generically these nets are BL to a lattice, and for some choices of dimensions and sections, they are generically BDD to a lattice. We also show the existence of such nets which are not BDD to a lattice.

1. Introduction

A *separated net* in $\mathbb{R}^d$ is a subset Y for which there are $0 < r < R$ such that any two distinct points of Y are at least a distance r apart, and any ball of radius R in $\mathbb{R}^d$ contains a point of Y. Separated nets are sometimes referred to as *Delone sets*. The simplest example of a separated net is a lattice in $\mathbb{R}^d$, and it is natural to inquire to what extent a given separated net resembles a lattice. To this end we define equivalence relations on separated nets: we say that Y_1, Y_2 are *bi-Lipschitz equivalent*, or *BL*, if there is a bijection $f : Y_1 \to Y_2$ which is bi-Lipschitz, i.e. for some $C > 0$,

$$\frac{1}{C}\|x - y\| \leq \|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in Y_1$; we say they are *bounded displacement*, or *BD*, if there is a bijection $f : Y_1 \to Y_2$ for which

$$\sup_{y \in Y} \|f(y) - y\| < \infty; \tag{1.1}$$

finally, we say they are *bounded displacement after dilation*, or *BDD*, if there is a $\lambda > 0$ such that Y_1 and λY_2 (the dilation of Y_2 by a factor λ) are BD. Clearly BD implies BDD, and it is not hard to show that for separated nets, BDD implies BL. Moreover it follows from the Hall marriage lemma (see Proposition 2.1) that all lattices are in the same BDD (hence BL) class, and in the same BD class if they have the same covolume. A fundamental result in this context was the discovery in 1998 (by Burago-Kleiner [7] and McMullen [16]) that there are

separated nets which are not BL to a lattice. In fact their arguments showed that there are uncountably many BL-inequivalent separated nets.

A simple way to construct separated nets is via an $\mathbb{R}^d$-action. Namely, suppose X is a compact space, equipped with a continuous action of $\mathbb{R}^d$. We denote the action by $\mathbb{R}^d \times X \ni (v, x) \mapsto v.x \in X$. Now given $x \in X$ and a subset $\mathcal{S} \subset X$, we can define the 'visit set'

$$Y = Y_{\mathcal{S},x} \stackrel{\mathrm{def}}{=} \{v \in \mathbb{R}^d : v.x \in \mathcal{S}\}. \tag{1.2}$$

It is quite easy (see §2.2) to impose conditions on $\mathcal{S}$ guaranteeing that Y is a separated net for all x. For example, this will hold if X is a k-dimensional manifold, $\mathcal{S}$ is a *Poincaré section* (i.e., an embedded submanifold of dimension $k - d$ everywhere transverse to orbits) and the $\mathbb{R}^d$-action is *minimal* (i.e. all orbits are dense).

The net Y obviously depends on the dynamical system X chosen. We will focus on what is perhaps the simplest nontrivial case, namely when $X = \mathbb{T}^k \stackrel{\mathrm{def}}{=} \mathbb{R}^k/\mathbb{Z}^k$ is the standard k-torus, and $\mathbb{R}^k$ acts linearly. That is, denoting $\pi : \mathbb{R}^k \to \mathbb{T}^k$ the standard projection, and letting $V \cong \mathbb{R}^d$ be a d-dimensional linear subspace of $\mathbb{R}^k$, the action is given by

$$v.\pi(\mathbf{x}) = \pi(v + \mathbf{x}). \tag{1.3}$$

In this context we will say that Y is a *toral dynamics separated net*, with *associated dimensions* (d, k). We will say that a section $\mathcal{S} \subset \mathbb{T}^k$ is *linear* if it is the image under π of a bounded subset of a $(k - d)$-dimensional plane transverse to V. We remark that the toral dynamics separated nets are intimately connected to the well-studied *cut-and-project* constructions of separated nets. We briefly discuss this connection in §2.3, and refer the reader to [17, 20, 3, 4] for more information.

Note that the separated net Y depends nontrivially on the choices of the subspace V, the section $\mathcal{S}$, and the orbit $V.x$. We will be interested in *typical* toral dynamical nets; e.g. this might mean randomly choosing the acting subspace V in the relevant Grassmannian variety, and/or the section $\mathcal{S}$ in a finite dimensional set of shapes such as parallelotopes, etc. We remark (see §2.2) that different choices of x do not have a significant effect on the properties of Y.

The constructions of [7, 16] were rather indirect, and left open the question of whether any of the nets constructed via toral dynamics is equivalent (in the sense of either BL or BDD) to a lattice. In [8], Burago and Kleiner addressed this issue, and showed that a typical toral dynamics separated net with associated dimensions (2,3) is BL to a lattice. We analyze the situations in arbitrary dimensions (d, k). Our first result shows that being BL to a lattice is quite common for toral dynamics nets:

Theorem 1.1. *For a.e. d-dimensional subspace $V \subset \mathbb{R}^k$, for any $x \in \mathbb{T}^k$, and any linear section $\mathcal{S}$ which is $k-d$ dimensionally open and bounded, and satisfies* $\dim_M \partial\mathcal{S} < k-d$*, the corresponding separated net is BL to a lattice.*

The assumptions on the section appearing in the statement are explained in §2.2. The notation $\dim_M$ signifies the upper Minkowski dimension, a notion we recall in §4. It would be interesting to know whether there is a toral dynamics separated net which is *not* BL to a lattice.

Our second result deals with the equivalence relation BDD. Here the situation is more delicate, and we have the following:

Theorem 1.2. *Consider toral dynamics nets with associated dimensions (d, k).*

(1) *If $(k+1)/2 < d \leq k$, then for almost every V, any $x \in \mathbb{T}^k$, and linear section $\mathcal{S}$ which is $k-d$ dimensionally open and bounded, and satisfies* $\dim_M \partial\mathcal{S} = k-d-1$*, the corresponding separated net is BDD to a lattice.*
(2) *For any $2 \leq d \leq k$, for almost every V, for any $x \in \mathbb{T}^k$ and any linear section $\mathcal{S}$ which is a box with sides parallel to $k-d$ of the coordinate axes, the corresponding net is BDD to a lattice.*
(3) *For almost every linear section $\mathcal{S} \subset B$ which is a parallelotope, there is a residual set of subspaces V for which the corresponding net is not BDD to a lattice.*

Our strategy of proof is inspired by [8, 21]. We use work of Burago-Kleiner [8] and Laczkovich [15] to relates the notions of BL and BDD to rates of convergence of some ergodic averages for our toral $\mathbb{R}^d$-action. This rate of convergence is studied via harmonic analysis on $\mathbb{T}^k$, and leads to the study of Diophantine properties of the acting subspace V. The connection between Diophantine properties of V and rates of convergence of ergodic averages on $\mathbb{T}^k$ is standard and well-studied in the literature on discrepancy, see e.g. [11]. However none of the existing results in the literature supplied the estimates we needed. Before stating our results in this direction, we introduce some notation.

We will use boldface letters such as $\mathbf{v}, \mathbf{x}$ to denote vectors in $\mathbb{R}^k$, and denote their inner product by $\mathbf{v} \cdot \mathbf{x}$. Let $V = \mathrm{span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_d)$. For $T > 0$ we set

$$B_T \stackrel{\text{def}}{=} \left\{ \sum a_i \mathbf{v}_i : \max_i |a_i| \leq T \right\}. \tag{1.4}$$

The notation $|A|$ denotes the Lebesgue measure of a measurable set A in $\mathbb{R}^k$ or $\mathbb{T}^k$. Given $U \subset \mathbb{T}^k$, $T \geq 0$ and $\mathbf{x} \in \mathbb{R}^k$ we set

$$N_T(U, \mathbf{x}) \stackrel{\text{def}}{=} \int_{B_T} \chi_U\left(\pi(\mathbf{x} + \mathbf{t})\right)\, d\mathbf{t}.$$

The reader should note that this notation suppresses the dependence of $N_T(U, \mathbf{x})$ on the choice of the subspace V as well as the basis $\mathbf{v}_1, \ldots, \mathbf{v}_d$. We will denote by $\|\mathbf{m}\|$ the sup-norm of a vector $\mathbf{m} \in \mathbb{R}^k$, and say that $\mathbf{v}$ is *Diophantine* if there are positive constants c, s such that

$$|\mathbf{m} \cdot \mathbf{v}| \geq \frac{c}{\|\mathbf{m}\|^s}\ , \quad \text{for all nonzero } \mathbf{m} \in \mathbb{Z}^k. \tag{1.5}$$

We will say that V is *Diophantine* if it contains a Diophantine vector.

By an *aligned box* in $\mathbb{T}^k$ we mean the image, under π, of a set of the form $[a_1, b_1] \times \cdots \times [a_k, b_k]$ (a box with sides parallel to the coordinate axes), where $b_i - a_i < 1$ for all i (so that π is injective on the box).

Theorem 1.3. *Suppose V is Diophantine. Then there are constants C and $\delta > 0$ such that for any $\mathbf{x} \in \mathbb{R}^k$, any $T > 1$, and any aligned box $U \subset \mathbb{T}^k$,*

$$\Big| N_T(U, \mathbf{x}) - |U||B_T| \Big| \leq CT^{d-\delta}. \tag{1.6}$$

We remark that under a stronger Diophantine assumption, which still holds for almost every subspace V, conclusion (1.6) can be strengthened, replacing $T^{d-\delta}$ with $(\log T)^{k+2d+\delta}$. See Proposition 7.2.

Given a basis $\mathcal{T} = (\mathbf{t}_1, \ldots, \mathbf{t}_k)$ of $\mathbb{R}^k$, we denote

$$r_{\mathcal{T}}(\mathbf{m}) \stackrel{\text{def}}{=} \prod_{i=1}^{k} \min \left(1, \frac{1}{|\mathbf{t}_i \cdot \mathbf{m}|} \right), \tag{1.7}$$

and say that $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are *strongly Diophantine (with respect to $\mathcal{T}$)* if for any $\varepsilon > 0$ there is $C > 0$ such that for any $M > 0$,

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|\mathbf{m}\| \leq M}} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|} \leq CM^{\varepsilon}. \tag{1.8}$$

We say that $U \subset \mathbb{T}^k$ is *a parallelotope aligned with $\mathcal{T}$* if there are positive $b_1, \ldots, b_k$, and $\mathbf{x} \in \mathbb{R}^k$, such that $U = \pi(\widetilde{U} + \mathbf{x})$, where

$$\widetilde{U} \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{d} a_i \mathbf{t}_i : \forall i,\ a_i \in [0, b_i] \right\},$$

and π is injective on $\widetilde{U} + \mathbf{x}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be the standard basis for $\mathbb{R}^k$.

Theorem 1.4. *Suppose $\mathcal{T} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a basis for $\mathbb{R}^k$ such that $\mathbf{v}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ for each $i = d+1, \ldots, k$, and $\mathbf{v}_1, \ldots, \mathbf{v}_d$ is strongly Diophantine with respect to $\mathcal{T}$. Then for any $\delta > 0$ and any K there is $C > 0$ such that for*

all $\mathbf{x} \in \mathbb{T}^k$*, and any* U *which is a parallelotope aligned with* $\mathcal{T}$*, with sidelengths bounded above by* K*, we have*

$$\left| N_T(U, \mathbf{x}) - |U||B_T| \right| \leq CT^{\delta}. \tag{1.9}$$

As above, we will show in Proposition 7.2 that there is a stronger Diophantine hypothesis, which still holds for almost every V, under which T^{δ} in (1.9) can be replaced by $(\log T)^{k+2d+\delta}$.

Justifying the terminology, we will see in §7 that a subspace with a strongly Diophantine basis is Diophantine. We will also see that almost every choice of V (respectively $\mathcal{T}$) satisfies the Diophantine properties which are the hypotheses of Theorem 1.3 (resp., Theorem 1.4). The conclusions of Theorems 1.1 and 1.2 hold for these choices.

Besides the cut-and-project method, another well-studied construction of a separated net is the *substitution system* construction, and results analogous to ours have appeared for separated nets arising via substitution systems in recent work of Solomon [21, 22] and Aliste-Prieto, Coronel and Gambaudo [1]. Briefly, it was shown in these papers that all substitution system separated nets are BL to lattices and many but not all are BDD to lattices. A particular case of interest is the *Penrose net* obtained by placing one point in each tile of the Penrose aperiodic tiling of the plane. As shown by de Bruijn [6], the Penrose net admits alternate descriptions via both the cut-and-project and substitution system constructions. Using the latter approach, Solomon [21] showed that that the Penrose net is BDD to a lattice.

1.1. **Organization of the paper.** In §2 we review basic material relating sections for minimal flows and separated nets, and the relation to cut-and-project constructions. In §3 we state the results of Burago-Kleiner and Laczkovich, and use these to connect the properties of the separated net to quantitative equidistribution statements for flows. In §4 we discuss Minkowski dimension and show how to approximate a section by aligned boxes if the Minkowski dimension of the boundary is strictly smaller than d. The main result of §5 is Theorem 5.1, which provides good approximations to the indicator function of parallelotopes in $\mathbb{T}^k$ by trigonometric polynomials. We believe this result will be helpful for other problems in Diophantine approximation and ergodic theory of linear toral flows. In §6 we deduce an Erdős-Turán type inequality from Theorem 5.1 and apply it to prove Theorems 1.3 and 1.4. In §7 we adapt arguments of W. Schmidt to show that our Diophantine conditions are satisfied almost surely, and deduce Theorem 1.1 and parts (1) and (2) of Theorem 1.2 in §8. In §9 we prove Theorem 1.2(3).

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2. Basics

2.1. **Bounded displacement.** We first recall the following well-known facts.

Proposition 2.1. *Any two lattices of the same covolume are BD to each other, and any two lattices are BDD to each other. Moreover, if $Y \subset \mathbb{R}^d$ is BDD to a lattice, and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear isomorphism, then $T(Y)$ is also BDD to a lattice.*

Proof. Suppose L_1 and L_2 are lattices of the same covolume λ, and define a bipartite graph G whose vertices are the points of $L_1 \cup L_2$, and $x_1 \in L_1$, $x_2 \in L_2$ are joined by an edge if $\|x_1 - x_2\| \le r_1 + r_2$, where r_i is the diameter of a compact fundamental domain for L_i. To verify the conditions of the Hall marriage lemma [13], let D_2 be a fundamental domain for L_2, so that $\mathbb{R}^d = \bigsqcup_{y \in L_2} y + D_2$, and let $A \subset L_1$ with $N \stackrel{\mathrm{def}}{=} \# A$. Let F denote the set of points in $\mathbb{R}^d$ which are within a distance r_1 from points of A. Then F contains at least N copies of a fundamental domain for L_1 so has volume at least $N\lambda$. Therefore F intersects at least N of the sets $\{y + D_2 : y \in L_2\}$. By the definition of G, if F intersects $y + D_2$ then y is connected to an element of A by an edge. This implies that the number of neighbors of A is at least N. By the marriage lemma there is a perfect matching in G, which gives our required bijection.

If the covolumes of L_1, L_2 are not the same, first apply a homothety to one of them to reduce to the previous case. Now suppose L is a lattice in $\mathbb{R}^d$ and $\phi : Y \to L$ is a bijection moving points a uniformly bounded amount, then $T \circ \phi \circ T^{-1}$ is a bijection $T(Y) \to T(L)$ and it moves points a bounded amount because T is Lipschitz. This proves the second assertion. □

2.2. **Sections and minimal actions.** A standard technique for studying flows was introduced by Poincaré. Suppose X is a manifold with a flow, i.e. an action of $\mathbb{R}$. Given an embedded submanifold $\mathcal{S}$ transverse to the orbits, we can study the return map to $\mathcal{S}$ along orbits, and in this way reduce the study of the $\mathbb{R}$-action to the study of a $\mathbb{Z}$-action. We will be interested in a similar construction for the case of an $\mathbb{R}^d$-action, $d > 1$. Namely, given a space X equipped with an $\mathbb{R}^d$-action, we say that $\mathcal{S} \subset X$ is a *good section* if there are bounded neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of 0 in $\mathbb{R}^d$, such that for any $x \in X$:

(i) there is at most one $u \in \mathcal{U}_1$ such that $u.x \in \mathcal{S}$.
(ii) there is at least one $u \in \mathcal{U}_2$ such that $u.x \in \mathcal{S}$.

These conditions immediately imply that the set $Y = Y_{\mathcal{S},x}$ of visit times defined in (1.2) is a separated net; moreover the parameters r, R appearing in the definition of a separated net may be taken to be the same for all $x \in X$, since they depend only on $\mathcal{U}_1, \mathcal{U}_2$ respectively.

The action is called *minimal* if there are no proper invariant closed subsets of X, or equivalently, if all orbits are dense. The following proposition shows that good sections always exist for minimal actions on manifolds:

Proposition 2.2. *Suppose X is a compact k-dimensional manifold equipped with a minimal $\mathbb{R}^d$-action, and suppose $\mathcal{S} \subset X$ is the image of an open bounded $\mathcal{O} \subset \mathbb{R}^{k-d}$ under a smooth injective map which is everywhere transverse to the orbits and extends to the closure of $\mathcal{O}$. Then $\mathcal{S}$ is a good section.*

Proof. Since $\mathcal{S}$ is transverse to orbits, for every $x \in \overline{\mathcal{S}}$ there is a bounded neighborhood $U = U_x$ of identity in $\mathbb{R}^d$ so that for $u \in U \smallsetminus \{0\}$, $u.x \notin \overline{\mathcal{S}}$. Since $\mathcal{O}$ is bounded, a compactness argument shows that U may be taken to be independent of x, and we can take $\mathcal{U}_1$ so that $\mathcal{U}_1 - \mathcal{U}_1 = \{x - y : x, y \in \mathcal{U}_1\} \subset U$, which immediately implies (i). Let

$$\widehat{\mathcal{S}} \stackrel{\mathrm{def}}{=} \{u.s : u \in U, s \in \mathcal{S}\}.$$

Then $\widehat{\mathcal{S}}$ is open in X. By a standard fact from topological dynamics (see e.g. [2]), the set of return times

$$\{u \in \mathbb{R}^d : u.x \in \widehat{\mathcal{S}}\}$$

is *syndetic*, i.e. there is a bounded set K such that for any $w \in \mathbb{R}^d$, there is $k \in K$ with $(w + k).x \in \widehat{\mathcal{S}}$. By minimality this implies that for any $x \in X$ there is $k \in K$ such that $k.x \in \widehat{\mathcal{S}}$. Taking $\mathcal{U}_2 = K - U$ we obtain (ii). $\square$

If X is not minimal, there will be some x and $\mathcal{S}$ for which Y is not syndetic. However good sections exist for any action:

Proposition 2.3. *For any action of $\mathbb{R}^d$ on a compact manifold, there are good sections.*

Proof. Fix a bounded symmetric neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^d$. We can assume that $\mathcal{U}$ is sufficiently small, so that for each $x \in X$ there is an embedded submanifold $\mathcal{S}_x$ of dimension $k - d$ such that the map

$$\mathcal{U} \times \mathcal{S}_x \to X, \quad (u, x) \mapsto u.x$$

is a diffeomorphism onto a neighborhood $\mathcal{O}_x$ of x. By compactness we can choose $x_1, \ldots, x_r$ so that the sets $\mathcal{O}_j = \mathcal{O}_{x_j}$ are a cover of X. By a small perturbation we can ensure that the closures of the $\mathcal{S}_j = \mathcal{S}_{x_j}$ are disjoint. Let

$\mathcal{S} = \bigcup_j \mathcal{S}_j$, then it is clear by construction that (ii) holds for $\mathcal{U}_2 = \mathcal{U}$. Since the $\mathcal{S}_j$ are disjoint, a compactness argument shows (i). $\square$

The following will be useful when we want to go from a section to a smaller one.

Proposition 2.4. *Suppose $\mathcal{S}$ is a section for an $\mathbb{R}^d$ action on a space X, $x \in X$, and $\mathcal{S} = \bigsqcup_{i=1}^r \mathcal{S}_i$ is a partition into subsets. Suppose that for $i = 1, \ldots, r$, each $Y_i \stackrel{\text{def}}{=} Y_{\mathcal{S}_i,x}$ is BD to a fixed lattice L. Then $Y_{\mathcal{S},x}$ is BDD to a lattice.*

Proof. Clearly $Y_{\mathcal{S},x} = \bigsqcup_1^r Y_i$, and by assumption, for each i there is a bijection $f_i : Y_i \to L$ moving points a bounded distance. Let $\hat{L}$ be a lattice containing L as a subgroup of index r and let $v_1, \ldots, v_r$ be coset representatives for $\hat{L}/L$. Then

$$f(y) = f_i(y) + v_i \quad \text{for } y \in Y_i$$

is the required bijection between Y and $\hat{L}$. $\square$

Proposition 2.5. *Suppose $\mathcal{S}_1, B$ are two good sections for an $\mathbb{R}^d$-action on a space X. Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_1', \mathcal{U}_2'$ be the corresponding sets as in (i) and (ii), for $\mathcal{S}_1$ and B respectively, and assume that*

$$\mathcal{U}_2' - \mathcal{U}_2' \subset \mathcal{U}_1. \tag{2.1}$$

Then there is $\mathcal{S}_2 \subset B$, a good section for the action, such that for each $x \in X$, the nets $Y_i = Y_{\mathcal{S}_i,x}$ as in (1.2) *($i = 1, 2$) are BD to each other.*

Proof. For each $x \in X$, let $u_x \in \mathcal{U}_2'$ be such that $u_x.x \in B$. Let $\mathcal{S}_2 \stackrel{\text{def}}{=} \{u_x.x : x \in \mathcal{S}_1\}$. First note that $\mathcal{S}_2$ is a good section: $\mathcal{U}_2'' \stackrel{\text{def}}{=} \mathcal{U}_2 + \{u_x : x \in \mathcal{S}_1\}$ satisfies (ii) for $\mathcal{S}_2$. Since $\mathcal{U}_1'$ satisfies (i) for B, it also satisfies (i) for $\mathcal{S}_2$.

Let $Y_i = Y_{\mathcal{S}_i,x}$ $(i = 1, 2)$. It remains to show that the Y_i are BD. For each $u \in Y_1$ we have $z = u.x \in \mathcal{S}_1$ so that $F(u).x \in \mathcal{S}_2$, where $F(u) = u + u_z$ and $u_z \in \mathcal{U}_2'$. Clearly F moves all points a bounded distance, and maps Y_1 to Y_2. We need to show that it is a bijection. If $u' \in Y_2$ then $u'.x = s_2 \in \mathcal{S}_2$, which implies that there is $s_1 \in \mathcal{S}_1$ with $s_2 = u_{s_1}.s_1$. This implies that $s_1 = (u' - u_{s_1}).x$ so that $u' - u_{s_1} \in Y_1$ satisfies $F(u' - u_{s_1}) = u' - u_{s_1} + u_{s_1} = u'$. Thus F is surjective. Now suppose $u_1, u_2 \in Y_1$ such that

$$u_1 + u_{z_1} = F(u_1) = F(u_2) = u_2 + u_{z_2},$$

where $z_i = u_i x \in \mathcal{S}_1$. Then by (2.1), $u_2 - u_1 = u_{z_1} - u_{z_2} \in \mathcal{U}_1$, so by (i) with $x = s_1$, we conclude that $u_2 = u_1$. $\square$

The nets $Y_{\mathcal{S},x}$ depend on the choice of x and $\mathcal{S}$. As Theorem 1.2 shows, different choices of $\mathcal{S}$ will lead to very different separated nets. However, as the following result shows, for much of our discussion the choice of x is immaterial.

Proposition 2.6. *Suppose X is a minimal dynamical system and $\mathcal{S}$ is a good section which is $(k-d)$-dimensionally open. If there is $x_0 \in X$ for which the separated net $Y_{\mathcal{S},x_0}$ is BD (resp. BDD, BL) to a lattice, then for every $x \in X$, the net $Y_{\mathcal{S},x}$ is also BD (resp. BDD, BL) to a lattice.*

Proof. We will prove the statement for the case of the BD equivalence relation, leaving the other cases to the reader.

Write $Y_0 \stackrel{\mathrm{def}}{=} Y_{\mathcal{S},x_0}$ and $Y \stackrel{\mathrm{def}}{=} Y_{\mathcal{S},x}$. Let $\mathcal{L} \subset \mathbb{R}^d$ be a lattice and let $f : Y_0 \to \mathcal{L}$ be a bijection satisfying

$$K \stackrel{\mathrm{def}}{=} \sup_{y \in Y} \|y - f(y)\| < \infty.$$

Let Ω be a compact fundamental domain for the action of $\mathcal{L}$ on $\mathbb{R}^d$, that is for each $z \in \mathbb{R}^d$ there are unique $\ell = \ell(z) \in \mathcal{L}$, $\omega = \omega(z) \in \Omega$ with $z = \ell + \omega$. Let $x \in X$ and let $u_n \in \mathbb{R}^d$ such that $u_n.x_0 \to x$. Using the continuity of the action on X, and the assumption that $\mathcal{S}$ is $(k-d)$-dimensionally open, it is easy to see that the translated nets $Y_0 - u_n$ converge to Y in the following sense. Let $B(x,T)$ denote the Euclidean open ball of radius T around x. For any $T > 0$ for which there is no element of Y of norm T, and any $\varepsilon > 0$ there is n_0 such that for any $n > n_0$, there is a bijection between $B(0,T) \cap Y$ and $B(0,T) \cap (Y_0 - u_n)$ moving points at most a distance ε.

Now for each k we take $n = n(k)$ large enough so that for each $y \in B(0,k) \cap Y$, there is $x = x(y) \in Y_0 - u_n$ with $\|y - x\| < 1$. Define $f_k : B(0,k) \cap Y \to \mathcal{L}$ by

$$f_k(y) \stackrel{\mathrm{def}}{=} f(x(y) + u_n) - \ell(u_n).$$

Then for each $k \geq k_0 > 0$, and each $y \in B(0,k) \cap Y$,

$$\begin{aligned} \|y - f_k(y)\| &\leq \|y - x(y)\| + \|x(y) + u_n - f(x(y) + u_n)\| + \|u_n - \ell(u_n)\| \\ &\leq 1 + K + \mathrm{diam}(\Omega); \end{aligned}$$

that is, points in $B(0,k_0) \cap Y$ are moved a uniformly bounded distance by the maps f_k, $k \geq k_0$. In particular the set of possible values of the maps $f_k(y)$, $k \geq k_0$ is finite. Thus by a diagonalization procedure we may choose a subset of the f_k so that for each $y \in Y$, $f_k(y)$ is eventually constant. We denote this constant by $\hat{f}(y)$. Now it is easy to check that $\hat{f}$ is a bijection satisfying (1.1). □

We now specialize to linear actions on tori. It is known that a linear action of a d-dimensional subspace $V \subset \mathbb{R}^k$ on $\mathbb{T}^k$ as in (1.3) is minimal if and only if V is *totally irrational*, i.e., not contained in a proper $\mathbb{Q}$-linear subspace of $\mathbb{R}^k$. Suppose V is totally irrational and of dimension d, so that the action of V on $\mathbb{T}^k$ is minimal. Note that when using this action to define separated nets via (1.2), one needs to fix an identification of V with $\mathbb{R}^d$; however, in light of

Proposition 2.1, for the questions we will be considering, this choice will be immaterial.

Let W be a subspace of dimension $k-d$, such that $\mathbb{R}^k = V \oplus W$. For any bounded open subset B' in W, such that $\pi|_{\bar{B}'}$ is injective, $B \stackrel{\text{def}}{=} \pi(B')$ is a good section, in view of Proposition 2.2. We do not assume that W is totally irrational, so that π need not be globally injective on W. We remind the reader that such sections will be called *linear* sections.

When discussing sections, there is no loss of generality in considering linear sections:

Corollary 2.7. *Let V and W be as above, and assume W is totally irrational. Then for any section $\mathcal{S}$ for the linear action of V on $\mathbb{T}^k$, there is a linear section $\mathcal{S}' \subset \pi(W)$ such that for any $x \in \mathbb{T}^k$, $Y_{\mathcal{S},x}$ and $Y_{\mathcal{S}',x}$ are BDD.*

Proof. Since W also acts minimally, for any $\varepsilon > 0$, there is a sufficiently large ball $B' \subset W$ such that $B = \pi(B')$ is ε-dense in $\mathbb{T}^k$. That is, we can make the neighborhood $\mathcal{U}_2$ appearing in (ii) as small as we wish. Thus, given any section $\mathcal{S}$ for the action of V, we can make B' large enough so that (2.1) holds. So the claim follows from Proposition 2.5. $\square$

When we say that the section $\mathcal{S}$ is $k-d$ dimensionally open, bounded, is a parallelotope, etc., we mean that $\mathcal{S} = \pi(\mathcal{S}')$ where $\mathcal{S}' \subset W$ has the corresponding properties as a subset of $W \cong \mathbb{R}^{k-d}$.

2.3. **Cut and project nets.** Fix a direct sum decomposition $\mathbb{R}^k = V \oplus W$ into $V \cong \mathbb{R}^d$, $W \cong \mathbb{R}^{d-k}$. Let $\pi_V : \mathbb{R}^k \to V$ and $\pi_W : \mathbb{R}^k \to W$ be the projections associated with this direct sum decomposition. Suppose $L \subset \mathbb{R}^k$ is a lattice, and $K \subset W$ is a non-empty bounded open set. The *cut-and-project construction* associated to this data is

$$\mathcal{N} = \mathcal{N}_{L,K,V,W} \stackrel{\text{def}}{=} \{x \in V : \exists y \in L, \pi_V(y) = x, \pi_W(y) \in K\}.$$

The set $\mathcal{N}$ is always a separated net in $V \cong \mathbb{R}^d$, and under suitable assumptions, is aperiodic (e.g. is not a finite union of lattices). This is a particular case of a family of more general constructions involving locally compact abelian groups. We refer to [20, 3, 4] for more details.

Unsurprisingly, the construction above may be seen as a toral dynamics separated net. Since we will not be using it, we leave the proof of the following to the reader:

Proposition 2.8. *Given L, $\mathbb{R}^k = V \oplus W$ and $K \subset W$ as above, there is a linear subspace $V' \subset \mathbb{R}^k$, a section $\mathcal{S} \subset \mathbb{T}^k$, and $x \in \mathbb{T}^k$, such that $\mathcal{N}_{L,K,V,W} = Y_{\mathcal{S},x}$, where $Y_{\mathcal{S},x}$ is as in* (1.2) *for the action* (1.3).

$\square$

3. Results of Burago-Kleiner and Laczkovich, and their dynamical interpretation

Let Y be a separated net. The question of whether Y is BL or BDD to a lattice is related to the number of points of Y in large sets in $\mathbb{R}^d$. More precisely, fix a positive number λ, which should be thought of as the asymptotic density of Y, and for $E \subset \mathbb{R}^d$, define

$$\mathrm{disc}_Y(E, \lambda) \stackrel{\mathrm{def}}{=} \Big| \# Y \cap E - \lambda |E| \Big|,$$

where $|E|$ denotes the d-dimensional Lebesgue measure of E ('disc' stands for *discrepancy*). If Y is a lattice, and E is sufficiently regular (e.g. a large ball), then one has precise estimates showing that $\mathrm{disc}_Y(E, \lambda)$ is small, relative to the measure of E. In this section we present some results which show that for arbitrary Y, bounds on $\mathrm{disc}_Y(E, \lambda)$ are sufficient to ensure that Y is BL or BDD to a lattice.

For each $\rho \in \mathbb{N}$ and $\lambda > 0$, let

$$D_Y(\rho, \lambda) \stackrel{\mathrm{def}}{=} \sup_B \frac{\mathrm{disc}_Y(B, \lambda)}{\lambda |B|},$$

where the supremum is taken over all cubes $B \subset \mathbb{R}^d$ of the form

$$B = [a_1\rho, (a_1+1)\rho] \times \cdots \times [a_d\rho, (a_d+1)\rho], \quad \text{with } a_1, \ldots, a_d \in \mathbb{Z}.$$

Theorem 3.1 (Burago-Kleiner)**.** *If there is $\lambda > 0$ for which*

$$\sum_\rho D_Y(2^\rho, \lambda) < \infty \tag{3.1}$$

then Y is BL to a lattice.

Proof. The theorem was proved in case $d = 2$ in [8], and in [1] for general d. □

Using this we state a dynamical sufficient condition guaranteeing that a dynamical separated net is BL to a lattice. We will denote the Lebesgue measure of $B \subset \mathbb{R}^d$ by $|B|$ and write the Lebesgue measure element as dt. Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a basis of $\mathbb{R}^d$ and define B_T via (1.4). Note that $|B_T| = CT^d$ for some $C > 0$. For $W \subset X$ and $x \in X$, denote

$$N_T(W, x) \stackrel{\mathrm{def}}{=} \int_{B_T} \chi_W(\mathbf{t}.x)\, d\mathbf{t},$$

where χ_W is the indicator function of W. The asymptotic behavior of such *Birkhoff integrals* as $T \to \infty$ is a well-studied topic in ergodic theory. The

action of $\mathbb{R}^d$ on X is said to be *uniquely ergodic* if there is a measure μ on X such that for any continuous function f on X, and any $x \in X$,

$$\left| \int_{B_T} f(\mathbf{t}.x)\, d\mathbf{t} - |B_T| \int_X f\, d\mu \right| = o(|B_T|).$$

We now show that a related quantitative estimate implies that certain dynamical nets are BL to a lattice.

Corollary 3.2. *Suppose $\mathbb{R}^d$ acts on X and $\mathcal{S}$ is a good section for the action. Let $\mathcal{U}_1$ be a neighborhood of identity in $\mathbb{R}^d$ satisfying (i) of §2.2, and let*

$$W \stackrel{\text{def}}{=} \{u.x : u \in \mathcal{U}_1,\ x \in \mathcal{S}\} \subset X. \tag{3.2}$$

Suppose there are positive constants a, C, δ such that for all $x \in X$ and $T > 1$,

$$\left| N_T(W, x) - a|B_T| \right| < C\, T^{d-\delta}. \tag{3.3}$$

Then for any $x \in X$, the net $Y_{\mathcal{S},x}$ as in (1.2) is BL to a lattice.

Proof. Let $x \in X$, $Y = Y_{\mathcal{S},x}$ and let $B = x' + B_T \subset \mathbb{R}^d$, i.e. B is a cube of side length $2T$, with sides parallel to the coordinate hyperplanes, and center at x'. We want to bound $\# Y \cap B$ in terms of $N_T(W, x')$. Let r denote the diameter of $\mathcal{U}_1$, and let $b = |\mathcal{U}_1|$. If $y \in Y \cap B$ then $y.x \in \mathcal{S}$ and hence $(y + u).x \in W$ for any $u \in \mathcal{U}_1$. This implies that

$$N_{T+r}(W, x) \geq (\# Y \cap B)\, b.$$

Similarly, if $\chi_W(y.x) = 1$ then there is $y' \in Y$ with $\|y' - y\| \leq r$, which implies that

$$N_{T-r}(W, x) \leq (\# Y \cap B)\, b.$$

Applying (3.3) we find that

$$\frac{a}{b}|B_{T-r}| - \frac{C}{b}(T - r)^{d-\delta} \leq \# Y \cap B \leq \frac{a}{b}|B_{T+r}| + \frac{C}{b}(T + r)^{d-\delta}.$$

So for any $\delta' < \delta$ there is T_0 such that for $T > T_0$, setting $\lambda = a/b$ gives

$$\mathrm{disc}_Y(B_T, \lambda) \leq T^{d-\delta'}.$$

Since $|B_T| = cT^d$ for some $c > 0$, we find that $D_Y(T, \lambda) = O(T^{-\delta'})$. From this (3.1) follows. $\square$

We now turn to analogous results for the relation BDD. Our results in this regard rely on work of Laczkovich. We first introduce some notation. For a measurable $B \subset \mathbb{R}^d$, we denote by $|B|$ the Lebesgue measure of B, by ∂B the boundary of B, and by $|\partial B|_{d-1}$ the $(d-1)$-dimensional volume of ∂B. By a *unit cube* (respectively, *dyadic cube*) we mean a cube of the form

$$[a_1, b_1) \times \cdots \times [a_k, b_k),$$

where for $i = 1, \ldots, k$ we have $a_i \in \mathbb{Z}$ and $b_i - a_i = 1$ (respectively, $b_i - a_i = 2^j$ for a non-negative integer j independent of i).

Theorem 3.3 ([15], Theorem 1.1)**.** *For a separated net $Y \subset \mathbb{R}^d$, and $\lambda > 0$, the following are equivalent:*

(1) *Y is BD to a lattice of covolume λ^{-1}.*
(2) *There is $c > 0$ such that for every finite union of unit cubes $\mathcal{C} \subset \mathbb{R}^d$,*

$$\mathrm{disc}_Y(\mathcal{C}, \lambda) \leq c\, |\partial\, \mathcal{C}|_{d-1}.$$

(3) *There is $c > 0$ such that for any measurable A,*

$$\mathrm{disc}_Y(A, \lambda) \leq c\, \left|(\partial\, A)^{(1)}\right|,$$

where $(\partial\, A)^{(1)}$ denotes the set of points at distance 1 from the boundary of A.

When applying this result, another result of Laczkovich is very useful. For sets $\mathcal{C}, Q_1, \ldots, Q_n$, we say that $\mathcal{C} \in S(Q_1, \ldots, Q_n)$ if $\mathcal{C}$ can be presented using $Q_1, \ldots, Q_n$ and the operations of disjoint union and set difference, with each Q_i appearing at most once. Then we have:

Theorem 3.4 ([15], Theorem 1.3)**.** *There is a constant κ, depending only on d, such that if $\mathcal{C}$ is a finite union of unit cubes in $\mathbb{R}^d$, then there are dyadic cubes $Q_1, \ldots, Q_n$, such that $\mathcal{C} \in S(Q_1, \ldots, Q_n)$ and for each j,*

$$\#\left\{i : Q_i \text{ has sidelength } 2^j\right\} \leq \kappa\, \frac{|\partial\, \mathcal{C}|_{d-1}}{2^{j(d-1)}}. \tag{3.4}$$

Corollary 3.5. *Suppose $\mathbb{R}^d$ acts on X and $\mathcal{S}$ is a good section for the action. Let $\mathcal{U}_1$ be a neighborhood of identity in $\mathbb{R}^d$ satisfying (i) of §2.2, and let W be as in* (3.2)*. Suppose there are positive constants a, C, δ such that for all $x \in X$ and $T > 1$,*

$$\Big| N_T(W, x) - a|B_T| \Big| < C\, T^{d-1-\delta}. \tag{3.5}$$

Then for any $x \in X$, the net $Y_{\mathcal{S},x}$ as in (1.2) *is BDD to a lattice.*

Proof. Let $b = |\mathcal{U}_1|$ and let $\lambda = a/b$. We verify condition (2) of Theorem 3.3. Arguing as in the proof of Corollary 3.2, we deduce from (3.5) that there are $0 < \delta' < \delta$, T_0 and C' such that for every cube Q of sidelength $T \geq T_0$ in $\mathbb{R}^d$,

$$\mathrm{disc}_Y(Q, \lambda) \leq C'\, T^{d-1-\delta'}. \tag{3.6}$$

By enlarging C' we can assume (3.6) holds for every $T \geq 1$. Given a finite union of unit cubes $\mathcal{C}$, let $Q_1, \ldots, Q_n$ be as in Theorem 3.4. Then we have:

$$\begin{aligned} \mathrm{disc}_Y(\mathcal{C}, \lambda) &\leq \sum_{i=1}^{n} \mathrm{disc}_Y(Q_i, \lambda) \\ &\overset{(3.4),(3.6)}{\leq} C'\kappa \sum_j \frac{|\partial\, \mathcal{C}|_{d-1}}{2^{j(d-1)}} 2^{j(d-1-\delta')} \\ &\leq \frac{C'\kappa}{1-2^{-\delta'}} \, |\partial \mathcal{C}|_{d-1}, \end{aligned}$$

as required. □

4. Minkowski dimension and approximation

Let $A \subset \mathbb{R}^k$ be bounded and let $r > 0$. We denote by $N(A, r)$ the minimal number of balls of radius r needed to cover A, and

$$\dim_M A \stackrel{\mathrm{def}}{=} \limsup_{r \to 0} \frac{\log N(A, r)}{-\log r}.$$

Equivalently (see e.g. [12, Chap. 3]), for $r > 0$ let $\mathcal{B}$ be the collection of boxes $[a_1, a_1 + r] \times \cdots \times [a_k, a_k + r]$ where the a_i are integer multiples of r, and let $S(A, r)$ denote the number of elements of $\mathcal{B}$ which intersect A. Then

$$\dim_M A = \limsup_{r \to 0} \frac{\log S(A, r)}{-\log r}.$$

From Theorem 1.3 we derive:

Corollary 4.1. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^k$ *be such that* $\mathrm{span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ *is Diophantine, and suppose* U *is a bounded closed set in* $\mathbb{T}^k$*, such that* $\dim_M \partial U < k$*. Then there are constants* C *and* $\delta > 0$ *such that for any* $\mathbf{x} \in \mathbb{R}^k$ *and any* $T > 1$*,*

$$\Big| N_T(U, \mathbf{x}) - |U||B_T| \Big| \leq C\, T^{d-\delta}.$$

Proof (assuming Theorem 1.3). Let K be a positive integer and for each $\mathbf{m} \in \mathbb{Z}^k$ let

$$C(\mathbf{m}) = \left[\frac{m_1}{K}, \frac{m_1+1}{K}\right] \times \cdots \times \left[\frac{m_k}{K}, \frac{m_k+1}{K}\right].$$

Define $A_1, A_2 \subset \mathbb{R}^k$ by

$$A_1 = \bigcup_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ C(\mathbf{m}) \subset U}} C(\mathbf{m}) \quad \text{and} \quad A_2 = \bigcup_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ C(\mathbf{m}) \cap U \neq \emptyset}} C(\mathbf{m}).$$

Clearly $N_T(A_1, \mathbf{x}) \le N_T(U, \mathbf{x}) \le N_T(A_2, \mathbf{x})$, so that

$$\Big|N_T(U, \mathbf{x}) - |U||B_T|\Big| \le \max_{i=1,2} \Big|N_T(A_i, \mathbf{x}) - |U||B_T|\Big|. \tag{4.1}$$

Now by the triangle inequality

$$\Big|N_T(A_1, \mathbf{x}) - |U||B_T|\Big| \le \Big|N_T(A_1, \mathbf{x}) - |A_1||B_T|\Big| + |B_T|\Big||A_1| - |U|\Big|. \tag{4.2}$$

The number of $\mathbf{m} \in \mathbb{Z}^k$ with $C(\mathbf{m}) \subset U$ is bounded above by a constant times M^k, so applying Theorem 1.3 to each of the aligned boxes $C(\mathbf{m})$ gives

$$\Big|N_T(A_1, \mathbf{x}) - |A_1||B_T|\Big| \le c_1 T^{d-\delta_0} K^k,$$

where c_1 and δ_0 are positive constants that are independent of K. Now our hypothesis on the dimension of the boundary guarantees that there is an $\varepsilon > 0$ such that the number of $\mathbf{m} \in \mathbb{Z}^k$ for which $C(\mathbf{m})$ intersects ∂U is bounded above by a constant times $K^{k-\varepsilon}$. Each of these boxes has volume K^{-k} and thus we have that

$$|B_T|\Big||A_1| - |U|\Big| \le c_2\, |B_T| \frac{K^{k-\varepsilon}}{K^k} \le c_3\, T^d K^{-\varepsilon},$$

with $0 < c_2 < c_3$ independent of K. Now we return to (4.2) and set $K = \lfloor T^{\delta_0/(k+\varepsilon)} \rfloor$ to obtain the bound

$$\Big|N_T(A_1, \mathbf{x}) - |U||B_T|\Big| \ \le c_1 T^{d-\delta_0} K^k + c_3 T^d K^{-\varepsilon} \le (c_1 + c_3) T^{d-\delta_0\varepsilon/(k+\varepsilon)}.$$

Setting $C = c_1 + c_3$, $\delta = \dfrac{\delta_0 \varepsilon}{k+\varepsilon}$ and applying the same argument to A_2 finishes the proof via (4.1). □

We now give a similar argument for bounded displacement.

Corollary 4.2. *Suppose* $d > (k+1)/2$ *and* $\mathcal{T} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ *is a basis of* $\mathbb{R}^k$ *satisfying the conditions of Theorem 1.4. Let* $\mathcal{S}$ *be a good section lying in a translate of* $\operatorname{span}(\mathbf{v}_{d+1}, \dots, \mathbf{v}_k)$*, which is closed in this affine subspace, and satisfies* $\dim_M \partial\mathcal{S} = k-d-1$*. Then we can choose* $\mathcal{U}_1$ *satisfying (i) of §2.2 so that, for the set* W *defined as in* (3.2)*, there are constants* C *and* $\delta > 0$ *such that for any* $\mathbf{x} \in \mathbb{R}^k$ *and any* $T > 1$,

$$\Big|N_T(W, \mathbf{x}) - |W||B_T|\Big| \le C\, T^{d-1-\delta}.$$

Proof (assuming Theorem 1.4). Much of this proof is analogous to the previous one, so to simplify the exposition we omit some of the notational details. We begin by covering the set $\mathcal{S}$ by $(k-d)$-dimensional boxes which are translates of aligned boxes in $\operatorname{span}(\mathbf{v}_{d+1}, \dots, \mathbf{v}_k)$ of sidelength $1/K$, $K \ge 1$,. As

before we construct disjoint unions A_1, A_2 of such boxes with the property that $A_1 \subset \mathcal{S} \subset A_2$, and we have that

$$\left|N_T(W, \mathbf{x}) - |W||B_T|\right| \le \max_{i=1,2} \left|N_T(A_i', \mathbf{x}) - |W||B_T|\right|,$$

with

$$A_i' \stackrel{\text{def}}{=} \{u.x : u \in \mathcal{U}_1,\ x \in A_i\}.$$

We choose $\mathcal{U}_1$ to be any parallelotope in $\mathbb{R}^d$ which satisfies (i) of §2.2, and which has sides parallel to $\mathbf{v}_1, \ldots, \mathbf{v}_d$. This is clearly possible since we can always replace our original choice of this set by any sub-neighborhood of the origin. With this choice of $\mathcal{U}_1$ our sets A_i' are unions of parallelotopes aligned with $\mathcal{T}$, with a uniform bound on their sidelengths. That is, parallelotopes to which Theorem 1.4 applies. The number of parallelotopes in A_1' is bounded above by a constant times K^{k-d}, so Theorem 1.4 tells us that for any $\delta_0 > 0$ there is a $c_1 > 0$ for which

$$\left|N_T(A_1', \mathbf{x}) - |A_1'||B_T|\right| \le c_1 T^{\delta_0} K^{k-d}.$$

Our hypothesis that $\dim_M \partial\mathcal{S} = k - d - 1$ leads to the inequality

$$|B_T|\left||A_1'| - |W|\right| \le c_2\, |B_T| \frac{K^{k-d-1}}{K^{k-d}} \le \frac{c_3 T^d}{K},$$

and using the triangle inequality as in (4.2) we have that

$$\left|N_T(A_1', \mathbf{x}) - |W||B_T|\right| \le c_1 T^{\delta_0} K^{k-d} + \frac{c_3 T^d}{K}.$$

Now using the hypothesis that $d > (k+1)/2$, we may assume that δ_0 has been chosen small enough so that there is a $\delta > \delta_0$ with $(1+\delta)(k-d) < (d-1-2\delta)$. Then setting $K = \lfloor T^{1+\delta} \rfloor$ we have that

$$\left|N_T(A_1', \mathbf{x}) - |W||B_T|\right| \le c_4 T^{d-1-\delta}.$$

Since the same analysis holds for A_2', the proof is complete. □

5. Trigonometric polynomials approximating parallelotopes

The proofs of Theorems 1.3 and 1.4 proceed with two major steps. The first step to prove an Erdős-Turán type inequality for Birkhoff integrals, and the second is to use Diophantine properties of the acting subspace to produce a further estimate on the error terms coming from the Erdős-Turán type inequality. Our goal in this section is to build up the necessary machinery to complete the first step.

Our approach to proving the Erdős-Turán type inequality requires approximations of the indicator function of a parallelotope by trigonometric polynomials which *majorize* and *minorize* it. To obtain the quality of estimates that

we need, we require the trigonometric polynomials to be close to the indicator function of the parallelotope in L^1-norm and to have suitably fast decay in their Fourier coefficients. The following theorem is the main result of this section, the Fourier analysis notation will be explained shortly.

Theorem 5.1. *Suppose that $U \subset \mathbb{R}^k$ is a parallelotope given by $U = L([-1,1]^k)$ for a linear isomorphism $L : \mathbb{R}^k \to \mathbb{R}^k$, such that $\pi|_U$ is injective. Let $\chi_U^{\mathbb{T}} : \mathbb{T}^k \to \mathbb{R}$ denote the indicator function of $\pi(U)$, and let $\mathcal{T}$ denote the basis $L(\mathbf{e}_i)$, $i = 1, \ldots, k$. Then for each $M \in \mathbb{N}$ there are trigonometric polynomials $\varphi_M(\mathbf{x})$ and $\psi_M(\mathbf{x})$ whose Fourier coefficients are supported in $\{\mathbf{m} \in \mathbb{Z}^k : \|L^{-t}\mathbf{m}\| \leq M\}$, where L^{-t} denotes the inverse-transpose of L, and*

$$\varphi_M(\mathbf{x}) \leq \chi_U^{\mathbb{T}}(\mathbf{x}) \leq \psi_M(\mathbf{x}) \tag{5.1}$$

for each $\mathbf{x} \in \mathbb{T}^k$. Moreover, there exists a constant $C > 0$, depending only on k, such that

$$\max\left\{|U| - \hat{\varphi}_M(\mathbf{0}), \hat{\psi}_M(\mathbf{0}) - |U|\right\} \leq \frac{C|\det L|}{M}, \tag{5.2}$$

and the Fourier coefficients of $\varphi_M(\mathbf{x})$ and $\psi_M(\mathbf{x})$ satisfy

$$\max\left\{\hat{\varphi}_M(\mathbf{m}), \hat{\psi}_M(\mathbf{m})\right\} \leq k3^{k+1}|\det L| r_{\mathcal{T}}(\mathbf{m}) \tag{5.3}$$

for all nonzero $\mathbf{m} \in \mathbb{Z}^k$, where $r_{\mathcal{T}}(\mathbf{m})$ is defined by (1.7)*.*

We note that some form of this result is alluded to in [14, Proof of Theorem 5.25], and since we could not find a suitable reference we will give the full details here. There are, however, known constructions which handle the case when U is rectangular [5, 9, 11]. Our proof requires the well known construction of Selberg regarding extremal approximations of the indicator functions of intervals by *entire functions*, which we will recall below. To move to several variables we bootstrap from the single variable theory using another construction due to Selberg, who never published his results. A similar construction can be found in [9].

Let $e(x) \overset{\text{def}}{=} \exp(2\pi i x)$. We will use the same notation for the Fourier transform of a function $F \in L^1(\mathbb{R}^N)$ and for a function $f : \mathbb{R}^N \to \mathbb{R}$ which is periodic with respect to $\mathbb{Z}^N$. That is

$$\hat{F}(\mathbf{t}) \overset{\text{def}}{=} \int_{\mathbb{R}^N} F(\mathbf{x}) e(-\mathbf{t}\cdot\mathbf{x})\, d\mathbf{x},\ \mathbf{t} \in \mathbb{R}^N;\ \ \hat{f}(\mathbf{m}) \overset{\text{def}}{=} \int_{[0,1)^N} f(\boldsymbol{\theta}) e(-\mathbf{m}\cdot\boldsymbol{\theta})\, d\boldsymbol{\theta},\ \mathbf{m} \in \mathbb{Z}^N.$$

The reader should have no difficulty disambiguating these two uses.

We are now in a position to define Selberg's functions. The interested reader will find a detailed account of what follows in the paper of Vaaler [24]. We

define the following auxiliary functions:

$$H(x) \stackrel{\text{def}}{=} \left(\frac{\sin(\pi x)}{\pi}\right)^2 \left(\sum_{n\in\mathbb{Z}} \frac{\operatorname{sgn}(n)}{(x-n)^2} + \frac{2}{x}\right),$$
$$K(x) \stackrel{\text{def}}{=} \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

Using the functions H and K above we define Beurling's functions

$$B, b : \mathbb{C} \to \mathbb{C}, \quad B(z) \stackrel{\text{def}}{=} H(z) + K(z) \text{ and } b(z) \stackrel{\text{def}}{=} H(z) - K(z).$$

For $M \in \mathbb{N}$ if we write

$$B_M(z) \stackrel{\text{def}}{=} B(Mz), \quad b_M(z) \stackrel{\text{def}}{=} b(Mz)$$

then it can be shown that

$$b_M(x) \leq \operatorname{sgn}(x) \leq B_M(x) \text{ for all } x \in \mathbb{R}.$$

Furthermore B_M and b_M have exponential type $2\pi M$ and

$$\|b_M - \operatorname{sgn}\|_{L^1(\mathbb{R})} = \|B_M - \operatorname{sgn}\|_{L^1(\mathbb{R})} = \frac{1}{M}.$$

Let

$$\chi(x) = \frac{\operatorname{sgn}(x+1) + \operatorname{sgn}(1-x)}{2}$$

be the indicator function of $[-1, 1]$, normalized to take the midpoint values at discontinuities. Then Selberg's functions, defined by

$$C_M(z) \stackrel{\text{def}}{=} \frac{B_M(z+1) + B_M(1-z)}{2}$$

and

$$c_M(z) \stackrel{\text{def}}{=} \frac{b_M(z+1) + b_M(1-z)}{2},$$

satisfy the inequalities

$$c_M(x) \leq \chi(x) \leq C_M(x).$$

Furthermore they have exponential type $2\pi M$ and

$$\|C_M - \chi\|_{L^1(\mathbb{R})} = \|\chi - c_M\|_{L^1(\mathbb{R})} = \frac{1}{M}. \tag{5.4}$$

Since C_M and c_M are entire functions of exponential type $2\pi M$ that are square integrable on the real axis, the Paley-Wiener theorem gives that their Fourier transforms $\hat{C}_M$ and $\hat{c}_M$ are supported in $[-M, M]$. Furthermore, by (5.4) we see that

$$\hat{C}_M(0) = 2 + \frac{1}{M} \text{ and } \hat{c}_M(0) = 2 - \frac{1}{M}. \tag{5.5}$$

We will need to bound the other Fourier coefficients:

Lemma 5.2. *For any positive integer M and real number ξ we have*

$$\max\left\{\left|\hat{C}_M(\xi)\right|, |\hat{c}_M(\xi)|\right\} \le 3\min\left\{1, \frac{1}{|\xi|}\right\}.$$

Proof. From (5.4) we have

$$\sup_{\xi\in\mathbb{R}}\left|\hat{C}_M(\xi) - \hat{\chi}(\xi)\right| \le \|C_M - \chi\|_{L^1(\mathbb{R})} = \frac{1}{M}.$$

In particular for any fixed ξ we have

$$\left|\hat{C}_M(\xi)\right| \le |\hat{\chi}(\xi)| + \frac{1}{M}. \tag{5.6}$$

For any $1 < |\xi| < M$ we have $|\hat{\chi}(\xi)| = |\sin(2\pi\xi)/\pi\xi| \le |\pi\xi|^{-1}$, hence

$$|\hat{\chi}(\xi)| + \frac{1}{M} < |\pi\xi|^{-1} + |\xi|^{-1} < \frac{2}{|\xi|},$$

therefore

$$|\hat{C}_M(\xi)| \le \frac{2}{|\xi|} \ \text{ for } \ 1 < |\xi| < M.$$

Recall that $\hat{C}_M(\xi) = 0$ if $|\xi| \ge M$ so it remains to show $|\hat{C}_M(\xi)| \le 3$ when $|\xi| < 1$. But by (5.6) we have

$$|\hat{C}_M(\xi)| \le \sup_{|\xi|<1}\left|\frac{\sin(2\pi\xi)}{\pi\xi}\right| + \frac{1}{M} \le 2 + 1.$$

This concludes the proof for $\hat{C}_M$, and the proof for $\hat{c}_M$ is nearly identical. □

5.1. **Majorizing and minorizing a cube in $\mathbb{R}^k$.** For any $M \in \mathbb{N}$ the indicator function $\chi_{[-1,1]^k}$ of $[-1,1]^k \subset \mathbb{R}^k$ is clearly majorized by the function

$$G_M(\mathbf{x}) \stackrel{\text{def}}{=} C_M(x_1)C_M(x_2)\cdots C_M(x_k). \tag{5.7}$$

Minorizing $\chi_{[-1,1]^k}$ requires a little more effort. For $i = 1, 2, \ldots, k$ define

$$L_M(\mathbf{x}; i) \stackrel{\text{def}}{=} c_M(x_i)\prod_{\substack{j=1\\ j\neq i}}^{k} C_M(x_j),$$

and then set

$$g_M(\mathbf{x}) \stackrel{\text{def}}{=} -(k-1)G_M(\mathbf{x}) + \sum_{i=1}^{k} L_M(\mathbf{x}; i).$$

We claim that

$$g_M(\mathbf{x}) \le \chi_{[-1,1]^k}(\mathbf{x}) \ \text{ for every } \mathbf{x} \in \mathbb{R}^k. \tag{5.8}$$

To establish this we use the following elementary inequality, which can be proved by induction on k:

$$\text{For any } \beta_1 \geq 1, \ldots, \beta_k \geq 1, \quad \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j\neq i}}^{k} \beta_j \leq 1 + (k-1) \prod_{j=1}^{k} \beta_j. \tag{5.9}$$

To verify the inequality (5.8), first suppose that $\mathbf{x} \notin [-1,1]^k$. Then there is an $1 \leq i \leq k$ with $|x_i| > 1$. Since $L_M(\mathbf{x}; i) \leq 0$ and $L_M(\mathbf{x}; j) \leq G_M(\mathbf{x})$ for all $j \neq i$, we have that

$$\sum_{i=1}^{k} L_M(\mathbf{x}; j) \leq (k-1) G_M(\mathbf{x}),$$

which implies $g_M(\mathbf{x}) \leq 0$. On the other hand if $\mathbf{x} \in [-1,1]^k$ then we have that

$$c_M(x_j) \leq 1 \leq C_M(x_j).$$

Then by (5.9) we have that

$$\sum_{i=1}^{k} c_M(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{k} C_M(x_j) \quad \leq \quad \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j\neq i}}^{k} C_M(x_j) \leq 1 + (k-1) \prod_{j=1}^{k} C_M(x_j),$$

and this together with the definition of g_M establishes (5.8).

5.2. **Proof of Theorem 5.1.** Define

$$\mathscr{G}_M(\mathbf{x}) \stackrel{\text{def}}{=} G_M \circ L^{-1}(\mathbf{x}) \text{ and } \mathscr{F}_M(\mathbf{x}) \stackrel{\text{def}}{=} g_M \circ L^{-1}(\mathbf{x}).$$

The results of §5.1 show that

$$\mathscr{F}_M(\mathbf{x}) \leq \chi_U(\mathbf{x}) \leq \mathscr{G}_M(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^k.$$

For the majorants and minorants of $\chi_U^{\mathbb{T}}$ define

$$\varphi_M(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{\mathbf{m} \in \mathbb{Z}^k} \mathscr{F}_M(\mathbf{x} + \mathbf{m}) \quad \text{and} \quad \psi_M(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{\mathbf{m} \in \mathbb{Z}^k} \mathscr{G}_M(\mathbf{x} + \mathbf{m}).$$

These functions are $\mathbb{Z}^k$ invariant, so we can view them as functions on $\mathbb{T}^k$, and since $\pi|_U$ is injective we have

$$\varphi_M(\mathbf{x}) \leq \chi_U^{\mathbb{T}}(\mathbf{x}) \leq \psi_M(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{T}^k. \tag{5.10}$$

To determine the Fourier transform of $\mathscr{G}_M$ and $\mathscr{F}_M$, observe that if $f : \mathbb{R}^k \to \mathbb{C}$ is an integrable function then $f \circ L^{-1}$ is also integrable and

$$\begin{aligned} \widehat{f \circ L^{-1}}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^k} f(L^{-1}\mathbf{x}) e(-\mathbf{x}^t \boldsymbol{\xi}) d\mathbf{x} \\ &= \int_{\mathbb{R}^k} f(\mathbf{y}) e(-\{L\mathbf{y}\}^t \boldsymbol{\xi}) |\det L| d\mathbf{y} \\ &= |\det L| \int_{\mathbb{R}^k} f(\mathbf{y}) e(-\mathbf{y}^t \{L^t \boldsymbol{\xi}\}) d\mathbf{y} \\ &= |\det L| \hat{f}(L^t \boldsymbol{\xi}). \end{aligned} \tag{5.11}$$

Since $\hat{G}_M(\boldsymbol{\xi}) = 0$ and $\hat{g}_M(\boldsymbol{\xi}) = 0$ when $\|\boldsymbol{\xi}\| \geq M$, both $\hat{\mathscr{F}}_M$ and $\hat{\mathscr{G}}_M$ are supported on $\{\boldsymbol{\xi} \in \mathbb{R}^k : \|L^{-t}\boldsymbol{\xi}\| \leq M\}$. Thus, by the Poisson summation formula and a classical theorem of Pólya and Plancherel [18], we have the following *pointwise* identities

$$\psi_M(\mathbf{x}) = \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ \|L^{-t}\mathbf{m}\| \leq M}} \hat{\mathscr{G}}_M(\mathbf{m}) e(\mathbf{m} \cdot \mathbf{x}) \tag{5.12}$$

and

$$\varphi_M(\mathbf{x}) = \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ \|L^{-t}\mathbf{m}\| \leq M}} \hat{\mathscr{F}}_M(\mathbf{m}) e(\mathbf{m} \cdot \mathbf{x}). \tag{5.13}$$

We will need the following formulas for the Fourier coefficients of ψ_M and φ_M:

$$\hat{\psi}_M(\mathbf{m}) = |\det L| \prod_{i=1}^k \hat{C}_M(\mathbf{t}_i \cdot \mathbf{m}) \tag{5.14}$$

and

$$\hat{\varphi}_M(\mathbf{m}) = |\det L| \left(-(k-1) \prod_{i=1}^k \hat{C}_M(\mathbf{t}_i \cdot \mathbf{m}) + \sum_{j=1}^k \hat{c}_M(\mathbf{t}_j \cdot \mathbf{m}) \prod_{\substack{i=1 \\ i \neq j}}^k \hat{C}_M(\mathbf{t}_i \cdot \mathbf{m}) \right). \tag{5.15}$$

To see (5.14), by (5.11) and basic properties of the Fourier transform we have that

$$\begin{aligned}\hat{\psi}_M(\mathbf{m}) &= \widehat{G_M \circ L^{-1}}(\mathbf{m})\\ &= |\det L|\hat{G}_M(L^t\mathbf{m})\\ &= |\det L|\prod_{i=1}^{k}\hat{C}_M(L^t\mathbf{m}\cdot\mathbf{e}_i)\\ &= |\det L|\prod_{i=1}^{k}\hat{C}_M(\mathbf{t}_i\cdot\mathbf{m}).\end{aligned}$$

The proof of (5.15) is similar. Now by using (5.14) and (5.15), together with (5.5) we find that

$$\hat{\psi}_M(\mathbf{0}) = |\det L|\prod_{i=1}^{k}\hat{C}_M(0) = |\det L|\left(2+M^{-1}\right)^k$$

and

$$\begin{aligned}\hat{\varphi}_M(\mathbf{0}) &= |\det L|\left(-(k-1)\prod_{i=1}^{k}\hat{C}_M(0)+\sum_{j=1}^{k}\hat{c}_M(0)\prod_{\substack{i=1\\ i\neq j}}^{k}\hat{C}_M(0)\right)\\ &= |\det L|\left(-(k-1)\left(2+\frac{1}{M}\right)^k + k\left(2-\frac{1}{M}\right)\left(2+\frac{1}{M}\right)^{k-1}\right)\\ &= |\det L|\left(2+\frac{1}{M}\right)^{k-1}\left(2-\frac{2k-1}{M}\right).\end{aligned}$$

The bounds (5.2) follow upon recalling that $|U| = 2^k|\det L|$. For the other Fourier coefficients we use Lemma 5.2 to obtain the inequalities

$$|\hat{\psi}_M(\mathbf{m})| = |\det L|\prod_{i=1}^{k}\left|\hat{C}_M(\mathbf{t}_i\cdot\mathbf{m})\right| \le 3^k|\det L|\prod_{i=1}^{k}\min\left\{1,\frac{1}{|\mathbf{t}_i\cdot\mathbf{m}|}\right\}$$

and

$$\begin{aligned}|\hat{\varphi}_M(\mathbf{m})| &\le |\det L|\left(\left|(k-1)\prod_{i=1}^{k}\hat{C}_M(\mathbf{t}_i\cdot\mathbf{m})\right| + \sum_{j=1}^{k}\left|\hat{c}_M(\mathbf{t}_j\cdot\mathbf{m})\prod_{\substack{i=1\\ i\neq j}}^{k}\hat{C}_M(\mathbf{t}_i\cdot\mathbf{m})\right|\right)\\ &\le 3^k(2k-1)|\det L|\prod_{i=1}^{k}\min\left\{1,\frac{1}{|\mathbf{t}_i\cdot\mathbf{m}|}\right\}.\end{aligned}$$

Combining these estimates with (5.10), (5.12), and (5.13) finishes our proof. □

6. An Erdős-Turán type inequality for Birkhoff integrals

From Theorem 5.1 we deduce:

Theorem 6.1. *Suppose that $d \le k$ are positive integers and that $V \subset \mathbb{R}^k$ is a subspace spanned by $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$. Let $\tilde{L} : \mathbb{R}^k \to \mathbb{R}^k$ be an affine isomorphism such that π is injective on the parallelotope $U = \tilde{L}([-1,1]^k)$. Let $\mathcal{T}$ denote the basis $L(\mathbf{e}_i), i = 1, \dots, k$, where L is the linear part of $\tilde{L}$. Then there is $C > 0$ (depending only on k) such that for any $M \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^k$ we have*

$$\Big| N_T(U, \mathbf{x}) - |U||B_T| \Big| \le C |\det L| \left(\frac{|B_T|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^{-t}\mathbf{m}\| \le M}} r_{\mathcal{T}}(\mathbf{m}) \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s}) \, d\mathbf{s} \right| \right). \tag{6.1}$$

Proof. If $\tilde{L}(\mathbf{y}) = L(\mathbf{y}) + \mathbf{y}_0$, we may replace $\mathbf{x}$ with $\mathbf{x} - \mathbf{y}_0$ to assume that $\tilde{L} = L$, so that Theorem 5.1 applies. For $M \ge 1$ we have from Theorem 5.1

$$\chi_U^{\mathbb{T}}(\mathbf{x}) - |U| \le \psi_M(\mathbf{x}) \le \frac{C' |\det L|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^{-t}\mathbf{m}\| \le M}} \hat{\psi}_M(\mathbf{m}) e(\mathbf{m} \cdot \mathbf{x}), \tag{6.2}$$

for some constant C' which depends only on k. By integrating both sides of (6.2) over $B_T - \mathbf{x}$ we find that

$$\begin{aligned} N_T(U, \mathbf{x}) - |U||B_T| &\le \left| \frac{C' |\det L| \, |B_T|}{M} + \int_{B_T} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^{-t}\mathbf{m}\| \le M}} \hat{\psi}_M(\mathbf{m}) e(\mathbf{m} \cdot \mathbf{s}) d\mathbf{s} \right| \\ &\le \frac{C' |\det L| \, |B_T|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^{-t}\mathbf{m}\| \le M}} \left| \hat{\psi}_M(\mathbf{m}) \right| \cdot \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s}) \, d\mathbf{s} \right| \\ &\overset{(5.3)}{\le} C |\det L| \left(\frac{|B_T|}{M} + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|L^{-t}\mathbf{m}\| \le M}} r_{\mathcal{T}}(\mathbf{m}) \cdot \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{s}) \, d\mathbf{s} \right| \right), \end{aligned}$$

where $C \overset{\text{def}}{=} \max(C', k3^{k+1})$. For a lower bound on $N_T(U, \mathbf{x}) - |U||B_T|$ we use $\varphi_M(\mathbf{x}) \le \chi_U^{\mathbb{T}}(\mathbf{x})$ in a similar way. □

Specializing to aligned boxes we obtain a generalization of the Erdős-Turán inequality.

Corollary 6.2. *Let the notation be as in Theorem 1.3. Suppose $U \subset \mathbb{R}^k$ is an aligned box. Then there is a positive constant C (depending only on k) such that for any $M \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^k$ we have that*

$$\Big|N_T(U,\mathbf{x}) - |U||B_T|\Big| \le C \left(\frac{|B_T|}{M} + \sum_{\substack{\mathbf{m}\in\mathbb{Z}^k \smallsetminus \{0\} \\ \|\mathbf{m}\| \le M}} r(\mathbf{m}) \left| \int_{B_T} e(\mathbf{m}\cdot\mathbf{t})\ d\mathbf{t} \right| \right), \quad (6.3)$$

where

$$r(\mathbf{m}) \stackrel{\text{def}}{=} \prod_{i=1}^{k} \min\left(1, \frac{1}{|m_i|}\right). \quad (6.4)$$

Proof. Note that an aligned box is the image of $[-1,1]^k$ under an affine map whose linear part L is of the form $\mathrm{diag}(\alpha_1,\ldots,\alpha_k)$ (since U is aligned) and with $0 < \alpha_i < 1/2$ (since π is injective on $L([-1,1]^k)$). This implies that $|\det L| < 1$ and $\|\mathbf{m}\| \le \|L^{-t}\mathbf{m}\|$ for any $\mathbf{m}$. Thus the statement follows from Theorem 6.1. □

We will need the following estimate for the integrals appearing on the right-hand-side of (6.1):

Proposition 6.3. *There is a constant C (depending only on d, k and the choice of Lebesgue measure on V) such that*

$$\left| \int_{B_T} e(\mathbf{m}\cdot\mathbf{s})\ d\mathbf{s} \right| \le C \prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|}. \quad (6.5)$$

Proof. For a constant C_1 depending on the choice of Lebesgue measure on V, we have:

$$\begin{aligned} \left| \int_{B_T} e(\mathbf{m}\cdot\mathbf{s})\ d\mathbf{s} \right| &= C_1 \prod_{i=1}^{d} \left| \int_{-T}^{T} e((\mathbf{m}\cdot\mathbf{v}_i)s_i)\ ds_i \right| \\ &= C_1 \prod_{i=1}^{d} \frac{|\sin(2\pi(\mathbf{m}\cdot\mathbf{v}_i)T)|}{\pi|\mathbf{m}\cdot\mathbf{v}_i|} \\ &\le \frac{C_1}{\pi^d} \prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|}. \end{aligned}$$

□

Proof of Theorem 1.3. Let $\mathbf{v}$ be a Diophantine vector in the subspace spanned by $\mathbf{v}_1, \dots, \mathbf{v}_d$, and write $\mathbf{v} = \sum_{i=1}^d x_i \mathbf{v_i}$. Fix c, s as in (1.5), and let $s' > s$. If $\mathbf{m} \in \mathbb{Z}^k$ satisfies

$$\max_{1 \le i \le d} |\mathbf{m} \cdot \mathbf{v}_i| \le \|\mathbf{m}\|^{-s'}$$

then, for all but finitely many $\mathbf{m}$,

$$|\mathbf{m} \cdot \mathbf{v}| \le \left(\sum_{i=1}^d |x_i| \right) \|\mathbf{m}\|^{-s'} \le c \|\mathbf{m}\|^s.$$

Thus for some $c_1 > 0$ we have

$$\max_{1 \le i \le d} |\mathbf{m} \cdot \mathbf{v}_i| \ge c_1 \|\mathbf{m}\|^{-s'} \quad \text{for all} \quad \mathbf{m} \in \mathbb{Z}^k. \tag{6.6}$$

We will apply Corollary 6.2 with

$$M = \lfloor T^\delta \rfloor, \quad \text{where } \delta = \frac{1}{d + s' + 1}. \tag{6.7}$$

Assume that the maximum in (6.6) is attained for $i = 1$. It follows that for some $c_4, c_3, c_2 > 0$,

$$\begin{aligned}
\left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{t}) \, d\mathbf{t} \right| &= c_2 \left| \int_{[-T,T]^d} e\left(\mathbf{m} \cdot \left(\sum t_i \mathbf{v}_i \right) \right) \, d\mathbf{t} \right| \\
&= c_2 \prod_{i=1}^d \left| \int_{-T}^T e((\mathbf{m} \cdot \mathbf{v}_i) t_i) \, dt_i \right| \\
&\le c_2 \, \frac{|\sin(2\pi(\mathbf{m} \cdot \mathbf{v}_1)T)|}{\pi |\mathbf{m} \cdot \mathbf{v}_1|} \, (2T)^{d-1} \\
&\le c_3 \, \frac{T^{d-1}}{|\mathbf{m} \cdot \mathbf{v}_1|} \overset{(6.6),(6.7)}{\le} c_4 \, T^{d-1+s'\delta}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{0 < \|\mathbf{m}\| \le M} r(\mathbf{m}) \left| \int_{B_T} e(\mathbf{m} \cdot \mathbf{t}) \, d\mathbf{t} \right| &\le c_4 \, T^{d-1+s'\delta} \sum_{0 < \|\mathbf{m}\| \le M} r(\mathbf{m}) \\
&\le c_4 \, M^d \, T^{d-1+s'\delta} \overset{(6.7)}{\le} c_5 \, T^{d-\delta}.
\end{aligned}$$

It is clear that the constants c_5, δ do not depend on U or $\mathbf{x}$. Thus the theorem follows from Corollary 6.2. □

Remark 6.4. *The proof shows that if V is Diophantine with corresponding constant s, then δ can be taken to be any number smaller than $\frac{d+1}{d+s+1}$.*

Proof of Theorem 1.4. For a fixed δ, let $\varepsilon = \delta/d$ and let C_1, C_2, C_3 be the constants C appearing in (1.8), (6.1) and (6.5) respectively. Let $L_0 : \mathbb{R}^k \to \mathbb{R}^k$ be the linear isomorphism mapping $\mathbf{e}_i$ to $\mathbf{v}_i$, $i = 1, \ldots, k$. Any U which is a parallelotope aligned with T is of the form $U = \pi \circ \widetilde{L}([-1,1]^k)$, where $\widetilde{L}$ is an affine isomorphism whose linear part L is of the form $L = L_0 \circ A$, and A is diagonal (since U is aligned with $\mathcal{T}$) and has entries bounded above (by our assumption on the sidelength). This implies that $|\det L|$ is bounded above for all such L, and the operator norm of $\|L^t\|$ is bounded above. In particular there is a constant λ, which depends on $\mathbf{v}_1, \ldots, \mathbf{v}_k$ but is independent of L, such that $\|\mathbf{m}\| \leq \lambda^{-1}\|L^{-t}\mathbf{m}\|$ for all $\mathbf{m}$. Hence

$$\{\mathbf{m} \in \mathbb{Z}^k : \|L^{-t}\mathbf{m}\| \leq M\} \subset \{\mathbf{m} \in \mathbb{Z}^k : \|\mathbf{m}\| \leq \lambda M\}.$$

Applying Proposition 6.3, we find that for any $M > 0$, the right hand side of (6.1) is bounded above by

$$C_2|\det L| \left(\frac{|B_T|}{M} + C_3 \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{0\} \\ \|\mathbf{m}\| \leq \lambda M}} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|} \right).$$

Now taking $M = \lfloor T^d \rfloor$, and using our strongly Diophantine hypothesis, gives the required bound, with $C = C_1C_2C_3|\det L|\lambda^{\delta/d}$. □

7. Diophantine approximation to subspaces

The main result of this section shows that the Diophantine properties stated in the introduction hold almost surely. More precisely, properties of d-tuples of vectors in $\mathbb{R}^k$ hold almost everywhere with respect to Lebesgue measure on $\times_1^d \mathbb{R}^k \cong \mathbb{R}^{kd}$, and properties of vector spaces hold almost everywhere with respect to the smooth measure class on the Grassmannian variety.

The fact that almost every vector is Diophantine is a standard exercise using the Borel-Cantelli Lemma — or see [10] for a stronger statement. For the extension to strongly Diophantine vectors, we employ some ideas of Schmidt [19]:

Proposition 7.1. *Almost every $\mathbf{v}_1, \ldots, \mathbf{v}_d$ is strongly Diophantine with respect to any basis $\mathcal{T} = (\mathbf{t}_1, \ldots, \mathbf{t}_k)$ for $\mathbb{R}^k$ having the property that for each $i \in \{d+1, \ldots, k\}$, there is a j for which $\mathbf{t}_i$ is a multiple of $\mathbf{e}_j$.*

Proof. Fix $\varepsilon > 0$, let $R_1, \ldots, R_d$ be cubes in $\mathbb{R}^k$ of sidelength 1, and for each $1 \leq i \leq d$ and $\mathbf{m} \in \mathbb{Z}^k \smallsetminus \{\mathbf{0}\}$ let

$$I_{R_i}(\mathbf{m}) \stackrel{\text{def}}{=} \int_{R_i} \frac{d\mathbf{v}}{|\mathbf{m} \cdot \mathbf{v}|(-\log \min(1/2, |\mathbf{m} \cdot \mathbf{v}|))^{1+\varepsilon}}.$$

We estimate this integral by using the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v})$, where

$$u_i = \mathbf{m} \cdot \mathbf{v}, \quad u_j = v_j \text{ for } 1 \le j \le k,\ j \ne i.$$

The Jacobian determinant of this transformation is $1/m_i$. If we write R_i' for the image of R_i in the $\mathbf{u}$ coordinate system then it is clear that for $j \ne i$ the u_j coordinates of two points in R_i' cannot differ by more than 1. Using this fact we have

$$I_{R_i}(\mathbf{m}) \le \frac{2}{m_i} \int_0^{1/2} \frac{du_i}{u_i |\log u_i|^{1+\varepsilon}} + \frac{1}{m_i (\log 2)^{1+\varepsilon}} \int_{1/2}^{1/2+m_i} \frac{du_i}{u_i} \le c_1 \frac{\log(m_i)}{m_i},$$

where c_1 depends only on ε. Thus we have

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_d=1}^{\infty} \frac{I_{R_1}(\mathbf{m}) \cdots I_{R_d}(\mathbf{m})}{(\log m_1)^{2+\varepsilon} \cdots (\log m_d)^{2+\varepsilon}} \le c_2$$

with c_2 depending on ε but not on $\mathbf{m}$. On interchanging the orders of integration and summation this implies that for almost every $(\mathbf{v}_1, \dots, \mathbf{v}_d) \in R_1 \times \cdots \times R_d$,

$$S(\mathbf{v}_1, \dots, \mathbf{v}_d) \stackrel{\text{def}}{=} \sum_{m_1=1}^{\infty} \cdots \sum_{m_d=1}^{\infty} \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i| (\log m_i)^{2+\varepsilon} (-\log \min(1/2, |\mathbf{m} \cdot \mathbf{v}_i|))^{1+\varepsilon}} \tag{7.1}$$

is finite and independent of $m_{d+1}, m_{d+2}, \dots, m_k \in \mathbb{Z}$. Since the location of the cubes $R_1, \dots, R_d$ was arbitrary, $S(\mathbf{v}_1, \dots, \mathbf{v}_d) < \infty$ for almost every $(\mathbf{v}_1, \dots, \mathbf{v}_d) \in (\mathbb{R}^k)^d$. By grouping together the choices for $m_{d+1}, \dots, m_k$, we obtain

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ 0 < m_1, \dots, m_k \le M}} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m} \cdot \mathbf{v}_i|}$$

$$\le C (\log M)^{k-d} (\log M)^{d(2+\varepsilon)} P(M) S(\mathbf{v}_1, \dots, \mathbf{v}_d), \tag{7.2}$$

where

$$P(M) = \prod_{i=1}^{d} \max_{1 \le m_1, \dots, m_k \le M} (-\log \min(1/2, |\mathbf{m} \cdot \mathbf{v}_i|))^{1+\varepsilon}.$$

In the inequality in (7.2) we are using the fact that for each $i \in \{d+1, \dots, k\}$, the quantity $\mathbf{t}_i \cdot \mathbf{m}$ is always a fixed multiple of m_j for some j.

By a standard application of the Borel-Cantelli Lemma, for almost every $\mathbf{v} \in \mathbb{R}^k$ there is a constant $c = c(\mathbf{v}) > 0$ such that

$$|\mathbf{m} \cdot \mathbf{v}| \ge \frac{c}{M^{2k}} \quad \text{for all} \quad \mathbf{m} \in \mathbb{Z}^k \quad \text{with} \quad 0 < \|\mathbf{m}\| \le M.$$

Thus for almost every $\mathbf{v}_1, \ldots, \mathbf{v}_d$ and for any $\delta > 0$ we have that (7.2) is bounded above by a constant times $(\log M)^{k+2d+\delta}$.

Finally we can estimate

$$\sum_{0<\|\mathbf{m}\|\leq M} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|}$$

by partitioning the sum into 2^k subsets of points $\mathbf{m}$, according to which components of $\mathbf{m}$ are 0. To each one of these subsets we may then apply the above arguments to obtain the required bound. □

As a corollary of our proof, the conclusions of Theorems 1.3 and 1.4 can be considerably strengthened, as follows.

Proposition 7.2. *For almost every $\mathbf{v}_1, \ldots, \mathbf{v}_d$, and any basis $\mathcal{T}$ as in Proposition 7.1, for any $\delta > 0$ there is $c > 0$ so that*

$$\sum_{0<\|\mathbf{m}\|\leq M} r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|} \leq c\,(\log M)^{k+2d+\delta}. \tag{7.3}$$

Under this condition, the error terms on the right hand sides of (1.6) *and* (1.9) *can be replaced by $C(\log T)^{k+2d+\delta}$.*

Proof. The bound (7.3) was already proved above. For the rest of the claim, take $M = T^d$ and use (7.3) and Proposition 6.3 in Theorem 6.1. □

To conclude this section we mention the following easy fact:

Proposition 7.3. *If $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are strongly Diophantine then each $\mathbf{v}_i$ is Diophantine.*

Proof. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are strongly Diophantine with respect to $\mathcal{T} = (\mathbf{t}_1, \ldots, \mathbf{t}_k)$, let $s > k+d-1$, and let $i \in \{1, \ldots, d\}$. Suppose by contradiction that there are infinitely many vectors $\mathbf{m} \in \mathbb{Z}^k$ so that $|\mathbf{m}\cdot\mathbf{v}_i| < \frac{1}{\|\mathbf{m}\|^s}$. If $\mathbf{m}$ is one such vector then setting $M = \|\mathbf{m}\|$ and using Cauchy-Schwarz we find, for each $j \neq i$,

$$|\mathbf{m}\cdot\mathbf{v}_j| \leq M\|\mathbf{v}_j\|.$$

Noting that $r_{\mathcal{T}}(\mathbf{m}) \geq \prod_{i=1}^{k} \frac{1}{\|\mathbf{t}_i\|\cdot\|\mathbf{m}\|}$ gives

$$r_{\mathcal{T}}(\mathbf{m}) \prod_{i=1}^{d} \frac{1}{|\mathbf{m}\cdot\mathbf{v}_i|} \geq M^{-k} \left(\prod_{i=1}^{k} \frac{1}{\|\mathbf{t}_i\|}\right) \left(\prod_{j\neq i} \frac{1}{M\|\mathbf{v}_j\|}\right) \|\mathbf{m}\|^s \geq CM^{s-k-d+1}.$$

This holds along a sequence of $M \to \infty$. However for some $\varepsilon > 0$ this contradicts (1.8). □

8. Proofs of Theorem 1.1 and 1.2(1),(2)

Proof of Theorem 1.1. Let V be a Diophantine subspace, and let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a basis for V. Let $\mathcal{S}$ be a linear section which is $(k-d)$-dimensionally open and bounded, with $\dim_M \partial\mathcal{S} < k-d$, let $\mathcal{U}_1$ be a closed ball around 0 in V, satisfying (i) of §2.2, and define W via (3.2). Then W is bi-Lipschitz equivalent to $\mathcal{U} \times \mathcal{S}$ and hence, by [12, Formulae 7.2 and 7.3], $\dim_M \partial W < k$. Thus the Theorem follows from Corollaries 3.2 and 4.1. □

Proof of Theorem 1.2(1). Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ satisfy the conclusion of Proposition 7.1, and for $i = d+1, \ldots, k$, let $\mathbf{v}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ such that $\mathcal{T} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a basis of $\mathbb{R}^k$. Also let $V = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$. We need to show that for any linear section $\mathcal{S}$ in a space L transverse to V, such that $\dim \partial S = k-d-1$, and any $\mathbf{x} \in \mathbb{T}^k$, the corresponding is BDD to a lattice. To this end we will apply Corollaries 3.5 and 4.2 . Let B be a ball in L such that π is injective on B, and sets $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfying conditions (i) and (ii) of §2.2 for B'. Also let $L' \stackrel{\mathrm{def}}{=} \mathrm{span}(\mathbf{v}_{d+1}, \ldots, \mathbf{v}_k)$, and let B' be a ball in L' such that π is injective on B'. Then B' is a good section, let $\mathcal{U}_1', \mathcal{U}_2'$ be the corresponding sets as in §2.2.

Suppose first that B is small enough so that (2.1) holds. Then we can assume with no loss of generality that $\mathcal{S}$ is contained in B'. This in turn shows that the hypotheses of Corollaries 4.2 and 3.5 are satisfied, and Y is BDD to a lattice.

Now suppose (2.1) does not hold. Then we can partition $\mathcal{S}$ into smaller sets $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(r)}$ with equal volume and $\dim_M \partial\mathcal{S}^{(i)} = k-d-1$, such that the corresponding sets $\mathcal{U}_1^{(i)}$ satisfy (2.1). Now repeating the previous argument separately to each $\mathcal{S}^{(i)}$, we see that the corresponding net is BD to a fixed lattice L. Note that the lattice is the same because each $\mathcal{S}_i$ has the same volume. Now the result follows via Proposition 2.4. □

Proof of Theorem 1.2(2). Suppose $\mathcal{S}$ is a box with sides parallel to the coordinate axes; that is, there is $J \subset \{1, \ldots, k\}$, $|J| = k-d$, such that $\mathcal{S}$ is the projection under π of an aligned box in the space $V_J \stackrel{\mathrm{def}}{=} \mathrm{span}(\mathbf{e}_j : j \in J)$. As above, we can use Proposition 2.4 to assume that π is injective on a subset of V_J covering $\mathcal{S}$. According to Proposition 7.1, for almost every choice of $\mathbf{v}_1, \ldots, \mathbf{v}_d$, the space $V = \mathrm{span}(\mathbf{v}_i)$ is strongly Diophantine with respect to the basis

$$\mathcal{T} \stackrel{\mathrm{def}}{=} \{\mathbf{v}_i : i = 1, \ldots, d\} \cup \{\mathbf{e}_j : j \in J\}.$$

As in the preceding proof, choose a neighborhood $\mathcal{U}_1$ of 0 in V satisfying property (i) of §2.2 which is a box. Then the set W defined by (3.2) is a parallelotope aligned with $\mathcal{T}$. According to Theorem 1.4, (3.5) holds, and we can apply Corollary 3.5. □

9. Irregularities of distribution

In this section we will fix $1 < d < k$ and let $\mathcal{G}$ denote the Grassmannian variety of d-dimensional subspaces of $\mathbb{R}^k$. We denote by $\mathcal{G}(\mathbb{Q})$ the subset of rational subspaces. We will fix a totally irrational $k-d$ dimensional subspace $W \subset \mathbb{R}^k$, and let $\mathcal{S}$ be the image under π of a subset of W which is open and bounded. There is a dense G_δ subset of $V \in \mathcal{G}$ for which $\mathcal{S}$ is a good section for the action of V on $\mathbb{T}^k$; indeed, by the discussion of §2.2, this holds whenever V and W are transverse to each other and V is totally irrational.

If $Q \in \mathcal{G}(\mathbb{Q})$ then any orbit $Q.\mathbf{x}$ is compact; further if Q is transverse to W then $Q.\mathbf{x} \cap \mathcal{S}$ is a finite set for every $\mathbf{x} \in \mathbb{T}^k$. We say that $\mathcal{S}$ and Q are *not correlated* if there are $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{T}^k$ such that

$$\#\left(Q.\mathbf{x}_1 \cap \mathcal{S}\right) = \#\left(Q.\mathbf{x}_1 \cap \overline{\mathcal{S}}\right) \neq \#\left(Q.\mathbf{x}_2 \cap \mathcal{S}\right) = \#\left(Q.\mathbf{x}_2 \cap \overline{\mathcal{S}}\right) \tag{9.1}$$

(here $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$). We say that $\mathcal{S}$ is *typical* if there is a dense set of $Q \in \mathcal{G}$ for which $\mathcal{S}$ and Q are not correlated.

It is not hard to find typical $\mathcal{S}$:

Proposition 9.1. *Let $r = k-d$ and let W be a totally irrational r-dimensional subspace of $\mathbb{R}^k$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis for W and for $\mathbf{a} = (a_1, \ldots, a_r) \in (0,1)^r, \mathbf{b} = (b_1, \ldots, b_r) \in (0,1)^r$ let*

$$P(\mathbf{a}, \mathbf{b}) \stackrel{\text{def}}{=} \pi\left(\left\{\sum_1^r t_i \mathbf{w}_i : t_i \in (a_i, a_i + b_i)\right\}\right).$$

Then the set of $(\mathbf{a}, \mathbf{b})$ for which $P(\mathbf{a}, \mathbf{b})$ is not correlated with any rational subspace, and hence typical, is of full measure and residual in $(0,1)^{2r}$.

Proof. It is enough to show that for a fixed Q, the set of $\mathbf{a}, \mathbf{b}$ for which $P(\mathbf{a}, \mathbf{b})$ is correlated with Q has zero measure and is a submanifold of dimension less than $2r$ in $[0,1]^{2r}$. To see this, define two functions $F, \overline{F}$ on $\mathbb{T}^k$, by

$$F(\mathbf{x}) \stackrel{\text{def}}{=} \#\left(Q.\mathbf{x} \cap \mathcal{S}\right), \quad \overline{F}(\mathbf{x}) \stackrel{\text{def}}{=} \#\left(Q.\mathbf{x} \cap \overline{\mathcal{S}}\right).$$

We always have $F(\mathbf{x}) \leq \overline{F}(\mathbf{x})$, and $F(\mathbf{x}) = \overline{F}(\mathbf{x})$ unless $Q.\mathbf{x}$ intersects the boundary of $\mathcal{S}$. So if (9.1) fails then $F(\mathbf{x})$ always has the same value, for the values of $\mathbf{x}$ for which $Q.\mathbf{x} \cap \partial\, \mathcal{S} = \varnothing$.

Note that the values of $F, \overline{F}$ are constant along orbits of Q. The space of orbits for the Q-action is itself a compact torus Q' of dimension r. Let $\pi' : \mathbb{T}^k \to Q'$ be the projection mapping a point to its orbit. The discussion in the previous paragraph shows that the requirement that $\mathcal{S}$ and Q are correlated is equivalent to the requirement that the interior of $\mathcal{S}$ projects onto a dense open subset of Q' with fibers of constant cardinality. Clearly this property is destroyed if we vary $\mathcal{S}$ slightly in the direction orthogonal to Q. More precisely, for any $\mathbf{a}$ and $\mathbf{b}$, there is a small neighborhood $\mathcal{U}$ such that which the set of

$\mathbf{a}', \mathbf{b}'$ in $\mathcal{U}$ for which (9.1) fails is a proper submanifold of zero measure. This proves the claim.

$\square$

By similar arguments one can show that almost every ball, ellipsoid, etc., is typical.

Proposition 9.2. *If $\mathcal{S}$ is a bounded open set whose boundary is of zero measure (w.r.t. the Lebesgue measure on the subspace W), and $\mathcal{S}$ is typical, then there is a dense G_δ subset of V for which, for every $\mathbf{x} \in \mathbb{T}^k$, the separated net $Y_{\mathcal{S},\mathbf{x}}$ is not BDD to a lattice.*

Proof. Let $Q_1, Q_2, \ldots$ be a list of rational subspaces in $\mathcal{G}(\mathbb{Q})$ such that $\mathcal{S}$ and Q_i are not correlated for each i, and $\{Q_i\}$ is a dense subset of $\mathcal{G}$. For each i let $\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(i)}$ be two points in $\mathbb{T}^k$ for which (9.1) holds. Since the linear action of subspaces on $\mathbb{T}^k$ is the restriction of the continuous natural $\mathbb{R}^k$-action, for any $\varepsilon > 0$ and any $T > 0$ we can find a neighborhood of Q_i in $\mathcal{G}$ consisting of subspaces V such that for any $v \in V$ with $\|v\| < T$, and any $\mathbf{x} \in \mathbb{T}^k$, the distance in $\mathbb{T}^k$ between $v.\mathbf{x}$ and $v'.\mathbf{x}$ is less than ε, where v' is the orthogonal projection of v onto Q_i. We will fix below a sequence of bounded sets $M_i \subset Q_i$ and denote by $M_i^{(V)}$ the preimage, under orthogonal projection $V \to Q_i$, of M_i. Using our assumption on $\mathcal{S}$, by perturbing $\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(i)}$ slightly we can assume that $q.\mathbf{x}_1^{(i)}$ and $q.\mathbf{x}_2^{(i)}$ are not in $\partial\mathcal{S}$ when $q \in M_i$. Since $\mathcal{S}$ is relatively open in W, this implies that there is an open subset $\mathcal{V}_i$ of $\mathcal{G}$ containing Q_i, such that for every $V \in \mathcal{V}_i$ and for $\ell = 1, 2$,

$$\#\left\{q \in M_i : q.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\right\} = \#\left\{v \in M_i^{(V)} : v.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\right\}. \tag{9.2}$$

Then

$$\mathcal{G}_\infty \stackrel{\mathrm{def}}{=} \bigcap_{i_0} \bigcup_{i \geq i_0} \mathcal{V}_i$$

is clearly a dense G_δ subset of $\mathcal{G}$, and it remains to show that by a judicious choice of the sequence M_i, we can ensure that for any totally irrational $V \in \mathcal{G}_\infty$, for any $\mathbf{x}$, and any positive λ, c, the separated net $Y_{\mathcal{S},\mathbf{x}}$ does not satisfy condition (3) of Theorem 3.3.

For any i let C_i be a parallelotope which is a fundamental domain for the action of the lattice $Q_i \cap \mathbb{Z}^k$ on Q_i. Specifically we let

$$C_i \stackrel{\mathrm{def}}{=} \left\{ \sum_{j=1}^{d} a_j \mathbf{q}_j : \forall j,\ 0 \leq a_j < \|\mathbf{q}_j\| \right\},$$

where $\mathbf{q}_1, \ldots, \mathbf{q}_d$ are a basis of $Q_i \cap \mathbb{Z}^k$. We claim that there are positive constants c_1, c_2, C (depending on i) and sets M_i which are finite unions of

translates of C_i, of arbitrarily large diameter, such that:

$$|M_i| \geq c_1 \operatorname{diam}(M_i)^d; \tag{9.3}$$

$$\left|(\partial M_i)^{(1)}\right| \leq C \operatorname{diam}(M_i)^{d-1} \tag{9.4}$$

(where, as before, $(\partial M_i)^{(1)}$ is the set of points at distance 1 from ∂M_i). Indeed, we simply take M_i to be dilations by an integer factor, of C_i around its center. Then each M_i is homothetic to C_i and (9.3) and (9.4) follow. Now let N_i be the number of copies of C_i in M_i. Then

$$\#\{q \in M_i : q.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\} = N_i \cdot \#\{q \in C_i : q.\mathbf{x}_\ell^{(i)} \in \mathcal{S}\} = N_i \cdot \# \left(Q.\mathbf{x}_\ell^{(i)} \cap \mathcal{S}\right)$$

and

$$|M_i| = N_i \cdot |C_i|,$$

which implies via (9.3) and (9.4) that for some constant c_2,

$$|(\partial M_i)^{(1)}| \leq c_2 N_i^{1-1/d}.$$

If we set

$$c_3 \stackrel{\text{def}}{=} \frac{\left|\# \left(Q.\mathbf{x}_2^{(i)} \cap \mathcal{S}\right) - \# \left(Q.\mathbf{x}_1^{(i)} \cap \mathcal{S}\right)\right|}{2},$$

then for any λ, there is $\ell \in \{1, 2\}$ such that for $\mathbf{x}' = \mathbf{x}_\ell^{(i)}$ we have

$$|\#(Q.\mathbf{x}' \cap \mathcal{S}) - \lambda |C_i|| \geq c_3,$$

and hence

$$\frac{|\#(M_i.\mathbf{x}' \cap \mathcal{S}) - \lambda|M_i||}{|(\partial M_i)^{(1)}|} \geq \frac{N_i\, |\#(Q\mathbf{x}' \cap \mathcal{S}) - \lambda|C_i||}{c_2 N_i^{1-1/d}} \geq \frac{c_3}{c_2} N_i^{1/d}.$$

So by choosing N_i large enough we can ensure that for any λ, and $\mathbf{x}'$ one of the $\mathbf{x}_\ell^{(i)}$, we have

$$|\#(M_i.\mathbf{x}' \cap \mathcal{S}) - \lambda|M_i|| \geq i\, |(\partial M_i)^{(1)}|. \tag{9.5}$$

Now fixing λ and c we choose $i > c$ and choose $\mathbf{x}'$ as above depending on λ. If $V \in \mathcal{V}_i$ is totally irrational then for any $\mathbf{x} \in \mathbb{T}^k$ there is a sequence $v_n \in V$ such that $v_n.\mathbf{x} \to \mathbf{x}'$. So we may replace $\mathbf{x}'$ with $\mathbf{x}$ and M_i with $v_n + M_i$ for sufficiently large n, and (9.5) will continue to hold. In light of (9.2), if Y is the net corresponding to V, $\mathcal{S}$ and $\mathbf{x}$, and $E \stackrel{\text{def}}{=} v_n + M_i$, then we have shown $\operatorname{disc}_Y(E, \lambda) > c|(\partial E)^{(1)}|$, and we have a contradiction to condition (3) of Theorem 3.3. ☐

Proof of Theorem 1.2(3). Immediate from Propositions 9.1 and 9.2. ☐

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